Optimization of Phase-Locked Loops With Guaranteed Stability

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Keywords: stability, phase-locked loop, multi-objective optimization

Abstract

Phase-locked loops (PLL) have found applications in many industrial applications such as communication and control systems. The key requirements are stability and loop performance in terms of signal-to-noise ratio and tracking errors. Here we present a two-step approach to PLL design. First, we present a Lyapunov approach to analyze the loop stability. The parameter range that can guarantee stability can be easily derived in the process. Second, we present a multi-objective optimization method that can search a set of values within the above range of parameters to achieve an optimal trade-off between loop bandwidth, transient and steady-state performance. Simulation results are contained to illustrate the performance of our procedure.

1. Introduction

The principle of phase-locked loops (PLL) is simple; it consists of a phase detector, a loop filter, and a voltage controlled oscillator [1] [3-4]. There are two key characteristics that make PLL widely used in communications systems and control applications. First, the bandwidth of PLL can be very small which means the signal-to-noise ratio can be improved in several orders of magnitude. Second, the PLL can automatically track signal frequency.

Due to the nonlinear nature of the phase detector, the PLL is basically a nonlinear system. For PLL to work in a wide range of conditions, the stability of PLL is very important. Conventional approaches to stability analysis included linear analysis, phase plane plots, rule of thumb, and simulation. All these methods have limited applications in low-order PLLs. If the order of PLL goes beyond three, it becomes almost impossible to analyze the stability of the system.

Lyapunov stability theory has been widely used in control systems, especially in nonlinear systems such as robotics, motors and spacecrafts. A Lyapunov function is essentially an energy function. The essential idea of Lyapunov approach is to find a Lyapunov function for the system and if it can be shown that the derivative of the Lyapunov function is negative, then the overall system will be stable. In this paper, we present an approach to stability analysis of second order PLLs since they have practical applications in many areas.

Although stability is most important in the successful applications of PLL, it only provides a range of parameter values which guarantee the stability. Trial-and-error and/or extensive simulations are still needed to fine tune the system parameters to achieve optimal performance. There are many performance objectives: bandwidth of PLL, signal-to-noise ratio, steady-state tracking error, rise time, overshoots, etc. Some of these objectives are conflicting. For example, in order to have a high signal-to-noise ratio, we need a lower bandwidth. However. a lower bandwidth implies slow rise time and hence poor tracking performance. To satisfy all these objectives manually requires a lot of simulations and experience. We herein propose a general multi-objective optimization method which can satisfy all the performance objectives simultaneously even in the presence of nonlinearity such as phase detector in the system.

The paper is organized as follows. Section 2 will describe the Lyapunov approach to stability analysis of a second-order. PLL. Section 3 will introduce the multi-objective optimization paradigm. A simulation example will be detailed in Section 4 to illustrate the performance of the proposed approach. Finally, some comments on future research directions will be included in Section ii.

2. Stability Analysis Using Lyapunov Function

2.1 Model of PLL

A general block diagram of an analog PLL is shown below.

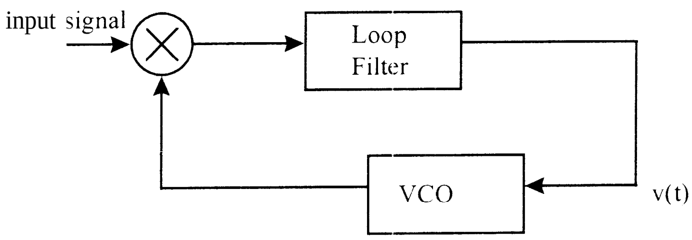

Fig. 1 An analog PLL.

Assume the input to the PLL is a sinusoid $\cos(\omega_i t + \phi)$ with ω_i the input frequency and ϕ the input phase and output of the VCO is $\sin(\omega_i t + \psi)$ with ψ the phase signal. A phase detector PD is just a multiplier which produces an output

$$e(t) = \cos(\omega_i t + \phi)\sin(\omega_i t + \psi)$$
$$= \frac{1}{2}\sin(\phi - \psi) + \frac{1}{2}\sin(2\omega_i t + \phi + \psi) . \quad (1)$$

The high-frequency component should be eliminated by using a low-pass filter. The cutoff frequency of the filter should be selected around ω_i. The reason is that we do not want to interfere with the function of the loop filter. The loop filter acts as controller to the PLL. Its function is to make sure the transient and steady-state response as good as possible. This filter is usually selected to have the form

$$G(s) = \frac{s+b}{s+a} \quad (2)$$

where a, b are design parameters which control the bandwidth of the loop. The output of the loop filter provides the control voltage $v(t)$ for the VCO. The VCO is a sinusoidal signal generator with an instantaneous phase given by

$$\omega_i t + \psi = \omega_i t + K\int_{-\infty}^{t} v(\tau)d\tau \quad (3)$$

with K a constant.

With a proper design of the low-pass filter to filter out the high-frequency component of the PD, we can therefore neglect the high-frequency component of the phase detector. Hence we can cast the PLL into a simpler and equivalent model shown in Fig. 2.

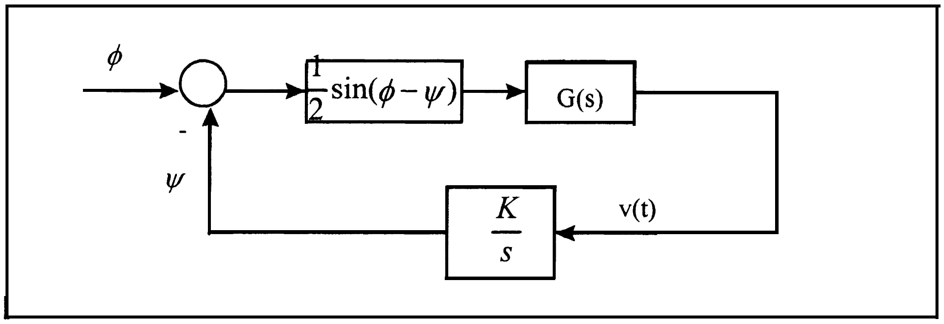

Fig. 2 Equivalent model of PLL.

2.2 Stability Analysis of PLL Using Lyapunov function

It should be emphasized that there is a nonlinearity in the system, i.e. the term $\sin(\phi - \psi)$. To analyze the stability, the conventional way is to linearize the nonlinear term by assuming $(\phi - \psi)$ is small. However, in many situation the phase error is not small due to either frequency error or phase error. To use Lyapunov approach, we first need to convert the system into a state-space model. Defining $\varepsilon = \phi - \psi$, we can write a model for the system in Fig. 2

$$\dot{x} = -ax + 0.5(b-a)\sin\varepsilon,$$
$$\dot{\varepsilon} = -Kx - 0.5K\sin\varepsilon. \qquad (4)$$

The state variable x denotes an internal state in the loop filter *G(s)*. Lyapunov approach is a method which can analyze the internal stability of the system, i.e. the stability of the states. If the states are stable, then the system will also be stable from the input-output point of view. The very first step of the analysis begins with the search of a Lyapunov function. This process depends heavily on one's experience. In [2], there are quite a number of examples on how to choose a proper Lyapunov function for a given system. For the PLL system, we choose the following Lyapunov function candidate,

$$V = (1-\cos\varepsilon) + \frac{1}{2}z^T P z \qquad (5)$$

where

$$z = \begin{bmatrix} x & \varepsilon \end{bmatrix}^T,$$
$$P = \begin{bmatrix} p_1 & p_2 \\ p_2 & p_3 \end{bmatrix}$$

is a positive definite matrix. V is a locally positive definite function (l.p.d.f.) since

(i) $V(z)=0$ if $z = 0$, and

(ii) $V(z) > 0$ if $z \neq 0$ for $z \in B$ where B is a ball around the equilibrium point $z = 0$.

There exists a well known theorem from nonlinear system analysis, which states the condition for system stability.

Theorem: The equilibrium point $z = 0$ is stable if there exists a continuously differentiable l.p.d.f. V such that

$$\dot{V}(z) \leq 0, \qquad \forall z \in B. \qquad (6)$$

Proof: See [2].

Thus our objective is to show that the derivative of I' is negative for the PLL system. Differentiating V in (5) gives

$$\dot{V} = \sin\varepsilon\dot{\varepsilon} + \frac{1}{2}\dot{z}^T P z + \frac{1}{2} z^T P \dot{z}$$
$$= [0.5(b-a)(p_1 - 0.5Kp_2 - K]x\sin\varepsilon + [0.5(b-a)(p_1 - 0.5Kp_2]\varepsilon\sin\varepsilon -$$
$$- (ap_1 + Kp_2)x^2 - (ap_2 + Kp_3)x\varepsilon \qquad (7)$$

Under the following conditions,

(1) $K > 0,\ b > a$ (8a)

(2) $0 > p_2 > -a/b$ (8b)

(3) $p_1 = \frac{K}{(b-a)}(p_2 + 1),\ p_3 = -ap_2 / K$ (8c)

it can be seen that

$$\dot{V} = -0.5K\sin^2\varepsilon - (ap_1 + Kp_2)x^2 + [0.5(b-a)p_2 - 0.5Kp_3]\varepsilon\sin\varepsilon$$
$$< 0 \qquad (9)$$

since

$$(ap_1 + Kp_2) > 0 \qquad (10)$$

and

$$[0.5(b-a)p_2 - 0.5Kp_3] < 0. \qquad (11)$$

Hence the PLL system is locally asymptotically stable. To extend the case for other types of second-order systems is not difficult and can be achieved following the same route as before.

3. Multi-Objective Optimization Technique for Optimal Loop Performance

The basic idea of the approach is to formulate the PLL design as a constrained optimization problem. Although there exist some approaches to optimization of PLL using Wiener filters, they only tradeoff the bandwidth with the total transient error. The approach that we propose to use is called multi-objective optimization method. It is more powerful than other methods because it can consider all specifications, including time domain and frequency domain specifications. It can also take in t0 account nonlinear effects of the system.

In PLL, the performance of tracking is judged based on many factors:

(a) steady-state error,
(b) transient response due to a step of phase,
(c) transient response due to a step of frequency,
(d) transient response due to a step of acceleration,
(e) loop behavior in the presence of an angle modulated input signal.

Besides these, another important criterion to judge the performance is the signal-to-noise ratio. These performance criteria are conflicting to one another. For example, if we want to have a fast transient response, we need a high loop gain. However, a high loop gain implies a high noise bandwidth which will reduce the signal-to-noise ratio. Our method can handle time-domain graphical constraints as well as nonlinear inequality constraints. One example of'graphical constraint is shown in Fig. 3 below.

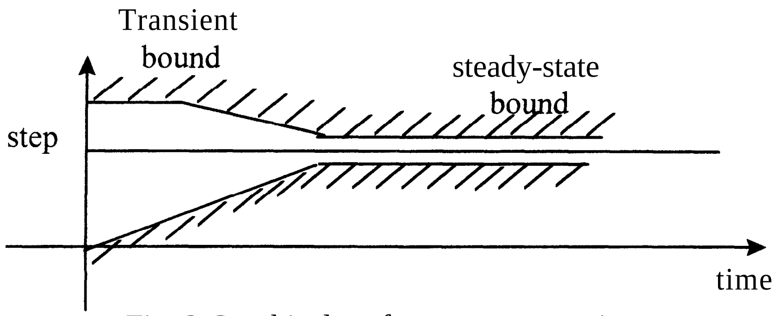

Fig. 3 Graphical performance constraints.

The transient performance is constrained by the transient bound shown in Fig. 3. In certain applications, percentage overshoot cannot be too large. The transient bound provides a upper limit for the overshoot. The steady-state bound is self-explanatory. Its purpose is to limit the steady-state tracking error.

The constraint can also be in the form of nonlinear inequality constraint. For example, if we do not want the noise bandwidth to exceed certain maximum values. We can write a constraint of the form [1]

$$B_L = \frac{1}{8} K \frac{0.5K + b}{0.5K + a} \leq 50 \; rad./s \qquad \text{for a second-order lag-lead loop filter} \qquad (12)$$

NOW all the stability and performance constraints can be translated into a set of performance objectives (J,(p), I = 1, 2, . . ., m)}, which are chosen such that if $J_i(p_1) < J_i(p_2)$ then the design parameter vector p_1 is better than the design parameterp $_2$, as far as the objective of J_i is concerned. This set of objectives can be considered to be a vector of objective functions, or a multi-objective function:

$$J(p) = \begin{bmatrix} J_1(p) \\ \cdots \\ J_m(p) \end{bmatrix}.$$

The performance specification is satisfied if J(p) < c for some positive vector c of "thresholds" i.e. if $J_i(p) < c_i$ for each i, and $c_i > 0$. To solve the above multiobjective function, there exists some procedures in the literature [5].

A typical design session proceeds as follows. First of all, we need to specify the constraints, the loop filter structure (active or lag-lead), the order of the loop filter, bandwidth of the PLL, etc. Then the simulation tool will simulate the performance through a few iterations. The software will compute values of the design parameters that improve the satisfaction of the constraints at each iteration. One can monitor the progress of the optimization by various graphical indicators and response plots.

4. Simulation Studies

We consider a second-order loop with a lag-lead compensator. The loop bandwidth should be less than 15 rad./s. The stability constraints are listed in Equations (8), (10), and (11). The settling time should be 0.1 second with respect to a step change of frequency of 50 rad./s. The initial parameters are

$$a = 1, \; b = 50, \; K = 50$$

After the optimization process, the optimal parameter values are:

$$a = 59.19, \; b = 126.55, \; K = 50.$$

In the optimization, we did not change the value of K since K is directly related to the bandwidth of the PLL.

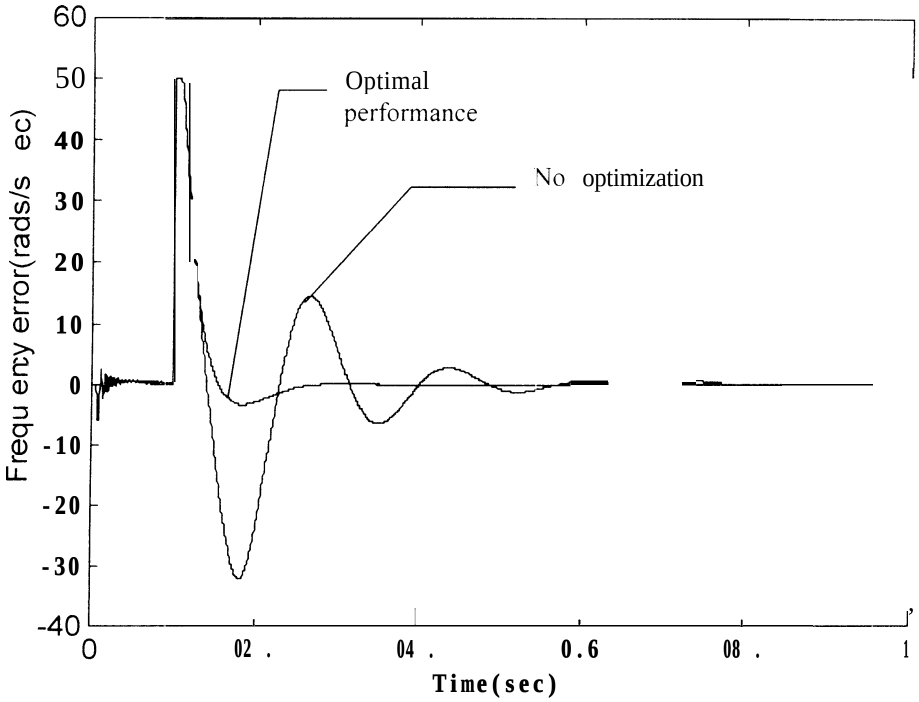

Fig. 4 Performance comparison of optimized and un-optimized PLL.

The performance comparison is shown in Fig. 4 which shows the un-optimized performance with the optimized performance. Even with the adjustment of two parameters, the performance can make quite a big difference.

5. Conclusions

A two-step procedure to the optimization of PLL with guaranteed stability is presented in this paper. The first step is to guarantee the loop stability by using the Lyapunov approach. In the process, parameter ranges (within the range the loop stability is -guaranteed) can be generated. The second step consists of a multi-objective optimization method that can choose a set of parameters within the previouslv mentioned parameter ranges to trade-off the loop bandwidth, tracking performance, and steady-state errors in an optimal fashion. Simulation results show that the proposed method indeed improves the performance of loop quite significantly.

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